<u>An Analysis on the Pedagogical Aspects of Number Theory to School Students</u>

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Problem 1

If p and $p^2 + 2$ are prime numbers, will $p^3 + 2$ also prime? If so, for how many primes p, will this be true?

Discussion

Since p and $p^2 + 2$ are primes so p can't be an even number i.e. 2

(otherwise $p^2 + 2$ won't be prime).

If we take p = 3 (prime) then $p^2 + 2 = 11$ (also prime) and $p^3 + 2 = 27 + 2 = 29$, which is also prime. Hence the statement is true for p = 3.

Now we need to explore if there are any other primes which satisfy this condition.

Let p be any prime number > 3 and can be written as $p = 3n + 1$ or $p = 3n + 2$.

<u>Case1</u>: If $p = 3n + 1$, then

$$p^2 + 2 = (3n + 1)^2 + 2$$

$$= 9n^2 + 6n + 3$$

$$= 3\,(3n^2 + 2n + 1), \text{ which is not a prime.}$$

Also $p^3 + 2 = (3n + 1)^3 + 2 = 27n^3 + 27n^2 + 9n + 1 + 2$

$$= 27n^3 + 27n^2 + 9n + 3$$

$$= 3\,(9n^3 + 9n^2 + 3n + 1), \text{ which is not a prime.}$$

<u>Case2</u>: If $p = 3n + 2$, then

$$p^2 + 2 = (3n + 2)^2 + 2$$

$= 9n^2 + 12n + 6 = 3(3n^2 + 4n + 2)$,

which is not a prime.

Also $p^3 + 2 = (3n + 2)^3 + 2$

$= 27n^3 + 54n^2 + 36n + 8 + 2$

$= 27n^3 + 54n^2 + 36n + 10$

$= 3(9n^3 + 18n^2 + 12n + 3) + 1$, this may be a prime.

Conclusion

Hence we can conclude that both p and $p^2 + 2$ are prime for only value p = 3 and so the statement is true for only value, p = 3, as $p^3 + 2 = 3^3 + 2 = 29$ which is also prime.

<u>Problem 2</u>

Do there exist natural numbers m and n such that $m^2 + (m + 1)^2 = n^4 + (n + 1)^4$?

Discussion

Let us assume that there exist natural numbers m and n such that

$$m^2 + (m + 1)^2 = n^4 + (n + 1)^4$$

Then

$$m^2 + m^2 + 2m + 1 = n^4 + n^4 + 4n^3 + 6n^2 + 4n + 1$$

$$\Rightarrow 2m^2 + 2m = 2n^4 + 4n^3 + 6n^2 + 4n$$

$$\Rightarrow m^2 + m = n^4 + 2n^3 + 3n^2 + 2n$$

Now add 1 to both the sides,

$$\Rightarrow m^2 + m + 1 = n^4 + 2n^3 + 3n^2 + 2n + 1$$

$$\Rightarrow m^2 + m + 1 = n^4 + n^2 + 1 + 2n^3 + 2n^2 + 2n$$

$$\Rightarrow m^2 + m + 1 = (n^2 + n + 1)^2 \quad \text{...........(1)}$$

Here right hand side is a perfect square, so left hand side should also be a perfect square. But

$$m^2 < m^2 + m + 1 < m^2 + 2m + 1$$

$$\text{or} \quad m^2 < m^2 + m + 1 < (m + 1)^2$$

This … g between two conse… not be a perfect square.

Conclusion

Hence our assumption is wrong and there doesn't exist two natural numbers m and n for which $m^2 + (m + 1)^2 = n^4 + (n + 1)^4$.

Problem 3

The four-digit number of the form (AABB) is a square of a number.

Find the possible values of A and B.

Discussion

Since it is given a four-digit number so A > 0 and B can be any digit between 0 and 9 (both inclusive).

The number (AABB) can be written as

$$1000A + 100A + 10B + B$$

$$= 1100A + 11B$$

$$= 11\ (100A + B)$$

Since the number (AABB) is a perfect square and 11 is one of its factor so

100A + B must also be a multiple of 11. Thus 100A + B must be equal to $11p^2$.

Also 100A + B must be a number of the form (A0B).

[For example if A = 2 and B = 9 so the number is 209]

Now for any three-digit number of the form (A0B) which is divisible by 11

then A – 0 + B must be a multiple of 11. In this case A + B = 11

Thus the number (AABB) can be written as

$$= 11\ (100A + B)$$

$$= 11\ (100A + 11 - A)$$

$$= 11\ (99A + 11)$$

$$= 11^2\ (9A + 1)$$

In order to be a complete square, 9A + 1 must be a perfect square, which is true for only value A = 7.

Hence the only possible four-digit number is 7744 which is equal to 88^2.

Conclusion

Here are the possible values of A and B satisfying the equation A + B = 11 in the following table. It is clear from the table that the only number which is a perfect square is 7744.

A	**B**	**(AABB)**	**(AABB) = p^2**
2	9	2299	
3	8	3388	
4	7	4477	
5	6	5566	
6	5	6655	
7	4	7744	88^2
8	3	8833	
9	2	9922	